

**Problem Set III: Due Thursday, October 29, 2015**

- 1.) A uniform bar of mass  $M$  and length  $2\ell$  is suspended from one end by a spring of force constant  $k$ . The bar can swing freely in one vertical plane. The spring moves only vertically. Derive the Hamiltonian and the Hamiltonian equations of motion.
- 2.) Consider the Helmholtz equation for a sound wave in a medium with index of refraction  $n(\underline{x})$ .

$$\nabla^2 \psi + \frac{\omega^2}{c_0^2} n(\underline{x})^2 \psi = 0.$$

- a.) For  $n(\underline{x})^2 = 1 + \delta(\underline{x})$ , where  $\delta \ll 1$ , and assuming sound is beamed in the  $\hat{z}$  direction, show the Helmholtz equation may be (approximately) re-written as:

$$2ik_z \frac{\partial \psi}{\partial z} + \nabla_{\perp}^2 \psi + \frac{\omega^2}{c_0^2} \delta(\underline{x}) \psi = 0.$$

- b.) Define  $k_r$  here. The above equation is called the parabolic wave equation. Discuss
  - i.) the approximations inherent to this formulation.
  - ii.) the physical meaning of the different terms.
  - iii.) the restrictions on  $\partial\psi/\partial z$ , etc.
- c.) Now, write  $\psi = A(\underline{x})e^{i\phi(\underline{x})}$ . Use the parabolic wave equation to derive coupled equations for phase  $\phi(\underline{x})$  and amplitude  $A(\underline{x})$ . Discuss the physical content of your result. Can you relate your result to that obtained using eikonal theory?

- d.) Extra Credit:

What happens if  $\delta(x)$  is stochastic, so  $\langle \delta(x)\delta(x') \rangle = \delta_0^2 c(x-x')$ . How would you calculate  $\psi$ ?

- 3.) Consider an ocean with sound speed a function of depth, i.e.  $c_s(z)$ , where  $z$  is depth as measured from the surface. What structure must  $c_s(z)$  have in order to act like an acoustic waveguide? Then using Fermat's Principle, determine the path that a ray takes to traverse a long distance  $\ell$ .

- 4a.) Consider the linearized acoustic equations

$$\frac{\partial \tilde{\rho}}{\partial t} + \underline{v} \cdot \nabla \tilde{\rho} = -\rho_0 \underline{\nabla} \cdot \tilde{\underline{v}}$$

$$\rho_0 \left( \frac{\partial \tilde{\underline{v}}}{\partial t} + \underline{v} \cdot \nabla \tilde{\underline{v}} \right) = -c_s^2 \nabla \tilde{\rho}$$

for an acoustic wave in a *flowing* medium, with flow velocity  $\underline{v}$ . This means that the frequency is Doppler shifted. Neglecting the spatial variation of the ambient flow in comparison to the wave length, derive the eikonal equation

$$\left( \frac{\partial \phi}{\partial t} + \underline{v} \cdot \nabla \phi \right)^2 = (\nabla \phi)^2 c_s^2(\underline{x}).$$

Show this is equivalent to the predictable:

$$(\omega - \underline{k} \cdot \underline{v})^2 = k^2 c_s^2(\underline{x}).$$

- b.) Now, derive the ray equations with flow.
- c.) Explain the physics of the ray equation  $\frac{dk}{dt} = -\frac{\partial}{\partial x}(\omega + \underline{k} \cdot \underline{v})$ . Use this to explain how a vertically sheared horizontal affects acoustic propagation. Why is it difficult to hear shouts from a distance on a windy day?
- d.) Show that the ray equations with flow are Hamiltonian.
- 5.) Consider a harmonic oscillator with distinct spring constants for the  $x, y, z$  directions. Derive and solve the Hamilton-Jacobi equation for this system.

- 6a.) What are the general conditions on the potential  $V(r, \phi, z)$  for separability of the Hamilton-Jacobi equation for particle motion in cylindrical coordinates?
- b.) Solve the Hamilton-Jacobi equation by separation, assuming  $V(r, \phi, z)$  has the requisite form.
- 7.) Derive an expression for the relationship between the unit normal vector to an acoustic path and the profile of the index of refraction. Relate this result to its counterpart for particle motion, using the equation for a particle path.
- 8.) On WKB: Consider the differential equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{Q(x)}{\epsilon^2} \psi = 0.$$

- a.) Use an ansatz of the form  $\psi \equiv \exp\left[\frac{i}{\epsilon} \sum_{n=0}^{\infty} \epsilon^n \phi_n(x)\right]$  to derive coupled equations for  $\phi_0$ ,  $\phi_1$  and  $\phi_2$ .
- b.) How does the  $\phi_0$  equation compare to the eikonal equation?
- c.) Solve for  $S_0$  and  $S_1$ , and use them to write an approximate form of  $\psi$ . Discuss your result.
- d.) What are the conditions of validity for the approximation strategy used here?
- 9.) Consider an acoustic wave propagating in a 3D medium with index of refraction  $n = n(x, y, z)$ .
- a.) For a short wavelength, constant frequency excitation, so that ray theory is applicable, discuss under what circumstances you will be able to solve the ray equations.
- b.) In the case of a.), calculate the eikonal phase  $\phi(\underline{x})$ .
- 10.) Determine the variation with altitude of the amplitude of a sound wave propagating in an isothermal atmosphere with gravity.